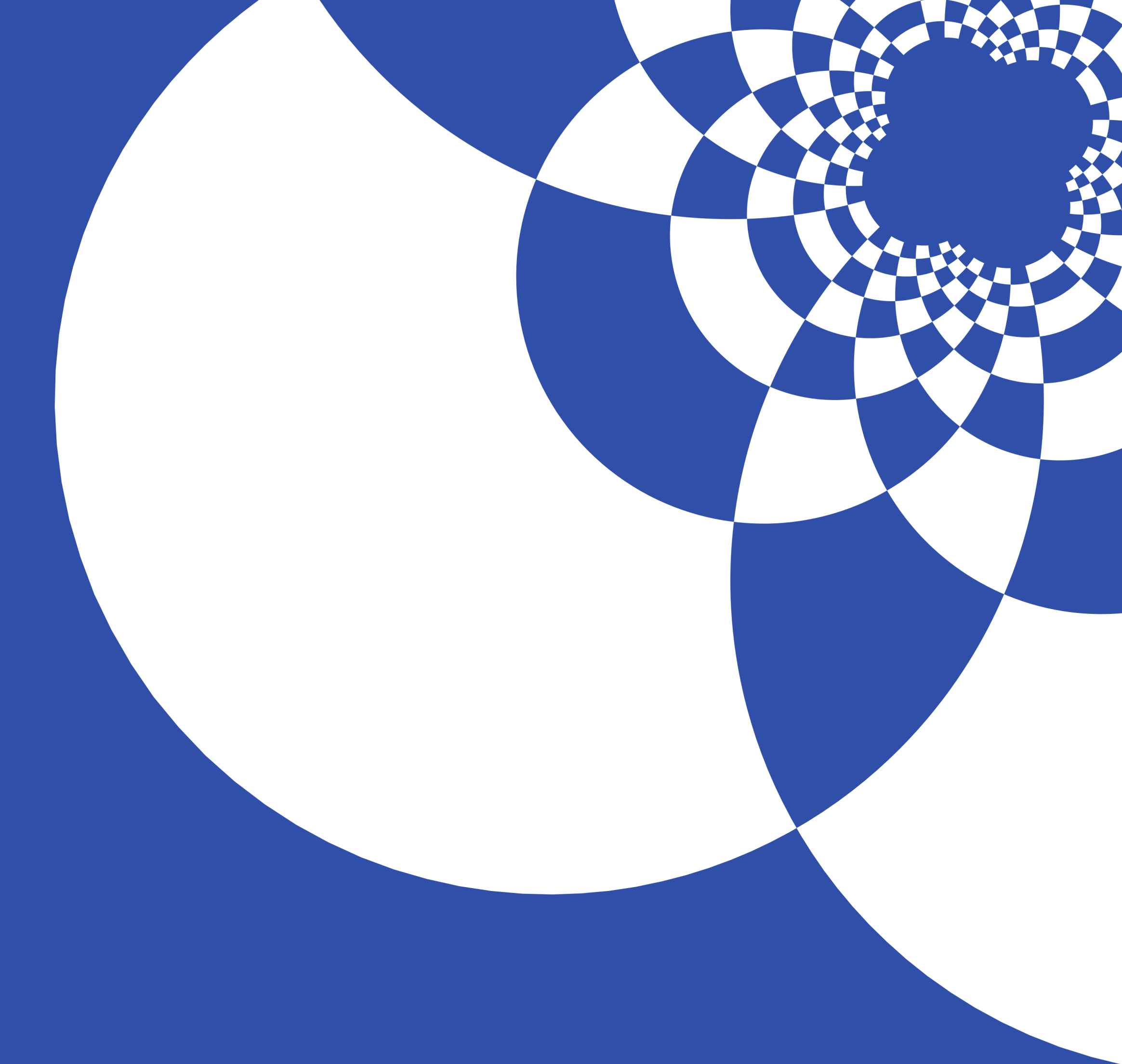

the invariant

TRINITY TERM 2024

Contents

Editorial

In spite of its title—picked, according to our Society's lore, from the volumes on Whitehead's bookshelf—our magazine has been anything but invariant. Since its birth in 1961, it has, like a stubborn Phoenix, risen anew in the hands of each generation of editors.

This year G. Zein, from Imperial, and Y. Zang, from Cambridge, have joined our ranks. G. J. Bala, already on to greater things, has once again used his pen to the benefit of his *alma mater*. Not to forget our very own T. Lam, S. Islam, T. L. Lee, N. Hayes, and Niphredil, who have each shared a unique piece, the discovery of which I leave you to enjoy.

Between their lines, you will find, in order of appearance: *Serioso*, R. Bauer; *Rythme no. 2*, R. Delaunay; *Tree of Knowledge no. 5*, H. af Klint; *Composizione*, 1916, P. Mondrian; *Ekstase*, K. Wiener; *Primordial Chaos, Group I, no. 16* and no. 7, H. af Klint; and *Forest Witches*, P. Klee.

It is with joy that I now pass on the torch—first lit sixty-three years ago by G. W. H. Smith, known to his readers by his humble signature: gwhs—to Toby Lam, our new editor, with whom it has been a pleasure to collaborate on this issue.

Yours invariably,
Diego Vurgait

As we reflect on the year past, we celebrated the 10th anniversary of The Andrew Wiles Building, which has been an excellent space for students, researchers and the wider public.

However, it has not been without challenges. We have lost Vicky Neale, who through her kindness and love for mathematics has inspired many. While preparing this magazine, I discovered that she was the editor of the 56th issue of *Eureka*, Cambridge's counterpart to *The Invariant*, 19 years ago. I sorely wish I could have sought her advice editing this magazine.

More recently, some of the proposed changes to Part C teaching have rattled undergraduates, prompting reassurances from the department. We hope that better days lie ahead.

Without the committee's support and contributions from our writers, this magazine could not exist. I am particularly grateful to Diego Vurgait, the outgoing magazine editor, for revitalising *The Invariant* and for his guiding hand. I am also indebted to Isaac Li for designing the cover as well as assisting with typesetting. Editing the magazine has been fascinating work and I have learnt much in the past months. I encourage any one of you who is interested to take up the post next year. As always, we are in great need of articles, essays, poems, puzzles, art and more for the next issue. Send in any ideas you have to `editor@invariants.org.uk` and we will read them with great interest.

Yours sincerely,
Toby Lam

Message from the President

Welcome to the 2024 edition of the Invariant. Featured in this magazine is a fantastic selection of articles, handpicked by our editors Diego and Toby: it explores topics including the Ising Model, the mathematics in art, and—a personal favourite—the lexicon of lecture notes. Suffice to say, everybody's interest will be piqued (though there won't necessarily be one article to pique everybody's interest!).

Among the many experiences coming to Oxford, the Invariants has stood out as a community where everyone can contribute—even a fresher like me! At the first event with the new committee—a meet and greet with puzzles and plenty of pizza—mathematicians from all colleges were gathered, working together and sharing laughs (and pizza). I also enjoyed seeing some biologists at our last talk; one of many examples of the multidisciplinarity of maths. This is my vision for the Invariants: a warm abode for all people to come together and enjoy mathematics. I've included a quote from Vicky Neale which I think exemplifies what is, and what will continue to be, the spirit of the Invariants.

Lastly, I would like to extend my deepest gratitude to Diego and Toby, without whom this edition would not be possible.

I'll see you all on page 68!

Yours,
Vasil Zelenkovski

Maths needs you

Even if you don’t know exactly where you’re heading,
you’re not maths-obsessed,
you don’t think you’re a genius or
you don’t fit the ‘stereotype’ of a maths student,
there’s a place for you in maths.

—Vicky Neale

The Inverse Function Theorem and the Legendre Transform

Toby Lam

If I gave you a monotone, differentiable function $f(x)$ with inverse $f^{-1}(x)$, could you give me the derivative and the antiderivative of $f^{-1}(x)$?

The first question is easy! We all know about the inverse function theorem. We just need to know $f^{-1}(x)$ and $f'(x)$. The second question, however, might bring up bad memories of your past self trying to remember calculus identities. It turns out that you only need to know $f^{-1}(x)$ and $\int f(x)\,dx$.

We will try to answer these two questions geometrically and show how they are related to each other. We will begin by proving the inverse function theorem geometrically.

A geometric proof of the inverse function theorem

As a reminder, we start off with a formal statement of the inverse function theorem.

Theorem (Inverse function theorem). *Given a continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ with $f'(a) \neq 0$ at some point a, there exists some interval I with a in its interior on which f has a continuously differentiable inverse f^{-1}, defined on $f(I)$, and with derivative*

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))} \ \forall x \in I.$$

In school, we proved the above algebraically by differentiating both sides of $f^{-1}(f(x)) = x$. In a course in analysis, we would be more rigorous.

We give the following geometric argument instead. We can first plot the function $y = f(x)$ on the Cartesian plane. Then we reflect the plot across the diagonal $y = x$. The reflected plot is exactly the plot of the inverse function. As all tangent lines of f are reflected over the diagonal, the slope of each tangent line is multiplicatively inversed. This is in essence the inverse function theorem, although we need to be careful keeping track of where each point gets reflected to.

Example. See figure 1. In red is the plot of $y = e^x$ and in blue is the plot of $y = \ln x$. We see how we they are related to each other by a reflection across the green diagonal line, $y = x$. The tangent line of e^x at $(-1, 1/e)$ is reflected across the diagonal to give the tangent line of $\ln x$ at $(1/e, -1)$. As such $\frac{d}{dx}\big|_{x=1/e} \ln x = 1/\frac{d}{dx}\big|_{x=-1} e^x = e$.

This suggests another way of thinking about the inverse function theorem. Recall that to plot a function $f(x)$, we are essentially parametrising a curve on $\mathbb{R}^2$ by sending x to $(x, f(x))$. We can find the slope of the curve at $(x, f(x))$ by finding the slope of the velocity vector $(1, f'(x))$ to the curve, which is exactly $f'(x)$.

Now, what if we parametrised the plot of the inverse function by sending x to $(f(x), x)$? The slope of the curve at $(f(x), x)$ will be the slope of the velocity vector $(f'(x), 1)$, which is $1/f'(x)$. This is exactly what the inverse function theorem says.

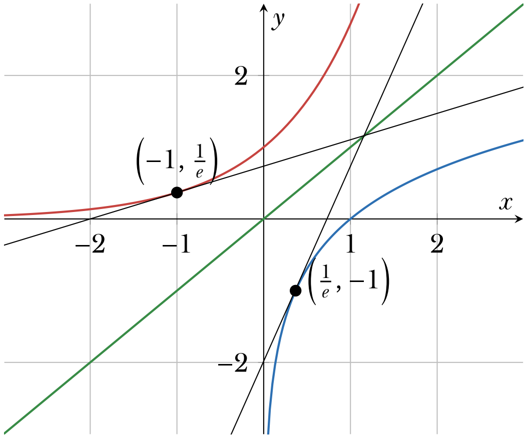

Figure 1. Plot of e^x and $\ln x$.

The Legendre transform

Now we try to answer the second question: what is the antiderivative of the inverse function? As for the inverse function theorem, we can approach it both algebraically and geometrically. While the algebraic method may not give us much insight, it is still worthwhile to mention. Almost all of the results in this section come from a 1905 paper by Laisant.

Allow me to pull something out of the hat. Let $F(x)$ be some antiderivative of $f(x)$. We can check that $xf^{-1}(x) - F(f^{-1}(x))$ is an antiderivative of $f^{-1}(x)$: By the product rule we have

$$\frac{d}{dx}xf^{-1}(x) = f^{-1}(x) + \frac{x}{f'(f^{-1}(x))}.$$

By the chain rule we have

$$\begin{aligned}\frac{d}{dx}F(f^{-1}(x)) &= \frac{f(f^{-1}(x))}{f'(f^{-1}(x))}\\ &= \frac{x}{f'(f^{-1}(x))}.\end{aligned}$$

Combining the two, we get

$$\frac{d}{dx}\left[xf^{-1}(x) - F(f^{-1}(x))\right] = f^{-1}(x).$$

For example, you can work out that $d/dx\,[x\ln(x) - x + C] = \ln x$ using the above. Unfortunately, it is not at all obvious where the formula comes from. One could reverse the logic by performing an integration by parts:

$$\begin{aligned}&\int f^{-1}(x)\,dx\\ &= xf^{-1}(x) - \int x\frac{d}{dx}f^{-1}(x)\,dx && \text{by parts}\\ &= xf^{-1}(x) - \int f(f^{-1}(x))\frac{d}{dx}f^{-1}(x)\,dx\\ &= xf^{-1}(x) - F(f^{-1}(x)) + C && \text{by FTC}\end{aligned}$$

for some constant C.

Perhaps the best argument is once again a geometric one. Laisant's formula tells us that if $f(x)$ is continuous and strictly increasing, then

$$\int_{f(a)}^{f(b)} f^{-1}(x)dx + \int_a^b f(x)dx = bf(b) - af(a).$$

A proof without words is presented in figure 2.

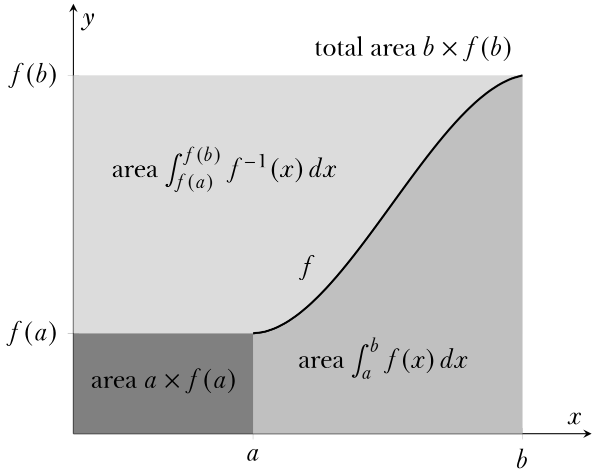

Figure 2. Visual proof of Laisant's formula.

We can connect Laisant's formula to the algebraic approach. Further assume that $f(x)$ is differentiable. Informally, consider the points $(a, f(a))$ and $(a+\epsilon, f(a+\epsilon))$ for some $a \in \mathbb{R}$ and some small $\epsilon > 0$. Laisant's formula tells us that

$$\int_{f(a)}^{f(a+\epsilon)} f^{-1}(x)\,dx + \int_a^{a+\epsilon} f(x)\,dx = (a+\epsilon)f(a+\epsilon) - af(a)$$

so

$$\begin{aligned} &G(f(a+\epsilon)) - G(f(a)) + F(a+\epsilon) - F(a) \\ &\qquad\qquad = (a+\epsilon)f(a+\epsilon) - af(a) \end{aligned}$$

for some antiderivative $G(x)$ of $f^{-1}(x)$. Dividing both sides by ϵ

and taking the limit $\epsilon \to 0$ we get

$$\left.\frac{d}{dx}\right|_{x=a} [G(f(x)) + F(x) - xf(x)] = 0.$$

As this is true for all $a \in \mathbb{R}$, we see that there is some constant C such that

$$G(f(x)) + F(x) - xf(x) = C.$$

Letting $\tilde{x} = f(x)$ and rearranging terms we see that

$$\tilde{G}(\tilde{x}) := \tilde{x}f^{-1}(\tilde{x}) - F(f^{-1}(\tilde{x}))$$

is an antiderivative of $f^{-1}(\tilde{x})$, which is what we expect.

Summary

In fact, taking $F(x)$ to $xf^{-1}(x) - F(f^{-1}(x))$ is called the Legendre transform, which is of immense significance in Lagrangian / Hamiltonian mechanics. I encourage the reader to explore this connection!

To summarise, we draw a commutative diagram

$$\begin{array}{ccc}
F(x) & \xrightarrow{\text{Legendre Transform}} & G(y) \\
\Big\downarrow \tfrac{d}{dx} & & \Big\downarrow \tfrac{d}{dy} \\
f(x) & \xrightarrow{\text{Taking the inverse}} & g(y) \; := f^{-1}(y) \\
\Big\downarrow \tfrac{d}{dx} & & \Big\downarrow \tfrac{d}{dy} \\
f'(x) & \xrightarrow{\text{Inverse Function Theorem}} & g'(y)
\end{array}$$

with the following relations.

$$\begin{aligned} &\text{Legendre:} && G(y) = y \times g(y) - F(g(y)) + C, \\ &\text{IFT:} && g'(y) = \frac{1}{f'(g(y))}. \end{aligned}$$

As such, the inverse function theorem is about what happens if you take the inverse then take the derivative. The Legendre transform is about what happens if you take the inverse then take the antiderivative.

References

Here is Laisant's original paper.

- C.-A. Laisant. *Intégration des fonctions inverses*. Nouvelles annales de mathématiques : journal des candidats aux écoles polytechnique et normale, 5 (1905), pp. 253-257. URL: `http://www.numdam.org/item/NAM_1905_4_5__253_0/`

Laisant's formula (and its curious connections with Young's inequality) is also exposed in Spivak's Calculus.

- Michael Spivak. *Calculus*. 4th ed. Publish or Perish, Inc., 2008. ISBN: 978-0-914098-91-1, Chapter 13, Problems 21 & 22, p. 276

The True Face of Hyperbolic Geometry

Gavin Jared Bala

The three geometries

Let us go back about 2300 years to ancient Greece and imagine Euclid at work in Alexandria. Perhaps, looking over his completed masterpiece—the *Elements*—he thought of the day he decided to ink the parallel postulate on his papyrus.

Though he knew it not, with that act he divided plane geometry into three parts, based on one difference:[1]

Given a line, and a point not on that line, one can draw through that given point...

1. *zero* lines parallel to that given line.
2. *exactly one* line parallel to that given line.
3. *more than one* line parallel to that given line.

The second case is Euclidean geometry, that of everyday life: the first that human knowledge grasped in its reach. Its natural model is any flat plane.

The first case is a little more exotic. It naturally arises while investigating solid geometry, being geometry on the surface of a sphere. Points are as they were, but lines are now great circles—geodesics on a sphere's surface—and they intersect in pairs of antipodal points.

[1]We state the parallel postulate as Playfair's axiom.

Moreover, between two antipodal points, there is no unique shortest path: any meridian suffices.[2]

This difference from the normal behaviour of lines led the ancients to reject this as a truly non-Euclidean geometry, unwilling to accept Hilbert's tables, chairs, and beer mugs in place of points, lines, and planes. Nonetheless, they studied spherical geometry in great detail, and its peculiarities are familiar to all who fly from Europe to the USA over Greenland. The first ancient Greek mathematical treatise to survive to our day is on this geometry: *On the Moving Sphere* by Autolycus of Pitane (c. 360–290 BC).

On a sphere of radius r, all triangles have an angle sum exceeding π. The larger the area, the larger their excess: the area of a spherical triangle with angles α, β and γ is exactly $r^2(\alpha + \beta + \gamma - \pi)$. The radius r is an absolute unit of length.

The third case is more vexed. Many investigators across centuries attempted to improve on Euclid and prove the parallel axiom a consequence of the other, less complicated ones. Spherical geometry could be excluded as lines were bounded: but this geometry posed a far sterner challenge to would-be refuters.

The braver ones gamely accepted the gambit, seeking a contradiction in vain, but proving wonderfully counterintuitive along the way. Particularly interesting are the results of Johann Heinrich Lambert (1728–1777):

1. The angles of a triangle add to *less* than π; and the larger the

[2]This issue can be cured by taking a quotient, identifying pairs of antipodal points, and studying elliptic geometry on the real projective plane instead.

area, the larger the defect, in linear proportion.

2. There is an absolute unit of length.
3. Two lines sharing a perpendicular diverge on either side of said perpendicular: thus, parallel lines are not equidistant. (In spherical geometry, they converge as all lines do.)

More such properties followed, and—strikingly to Lambert's eyes—they tended to resemble analogous well-known properties of spherical geometry, suggesting that there was a third geometry just over the horizon. And yet, he was still unsure: his work trails off, unfinished, after starting another refutation attempt.

The happy ending is well-known: as was said by William Kingdom Clifford, geometry found its Copernicus to match Euclid's Ptolemy. In 1829-1830, Nikolai Lobachevsky's work on hyperbolic geometry was published. In 1832, the independent work of János Bolyai appeared: as he wrote to his father when he discovered it in 1823, 'out of nothing I have created a strange new universe'. We cannot forget Carl Friedrich Gauss, who coined the very name of non-Euclidean geometry, yet did not publish.

Nowadays, the third geometry is known as hyperbolic (or Lobachevskian) geometry, and is a standard topic in an undergraduate mathematics curriculum. The three geometries are the cases of constant positive, zero, and negative curvature.

So rises the glorious edifice of mathematics. Yet one vexation remains: a 'homeland' for hyperbolic geometry is missing. Euclidean geometry has the plane, and spherical geometry has the sphere: but it is a sad theorem that the complete hyperbolic plane cannot be embedded in $\mathbf{R}^3$. We are reduced to studying models: the Poincaré disc model, the Poincaré half-space model, the Beltrami–Klein model—yet this is like studying spherical geometry without ever seeing a sphere, only various map projections. What then is the true face of hyperbolic geometry?

Imaginary spheres

That face was known to Lambert himself![3]

In his musing, Lambert noticed that the formulae of hyperbolic geometry can be derived from those of spherical geometry—*if one boldly posits that the sphere has imaginary radius!*

Consider the angular defect. On a sphere of radius r, the area of a triangle with angles α, β and γ is $r^2(\alpha + \beta + \gamma - \pi)$. Substituting $r = i$ gives us $\pi - (\alpha + \beta + \gamma)$—precisely the area of a *hyperbolic* triangle, if the constant negative curvature is -1!

The circumference and area of a circle of radius ρ in $\mathbf{S}^2$ (the unit sphere) are respectively $2\pi \sin \rho$ and $\pi \sin^2 \rho$; in $\mathbf{H}^2$ they are respectively $2\pi \sinh \rho$ and $\pi \sinh^2 \rho$.

The surface area and volume of a sphere of radius ρ in $\mathbf{S}^2$ are respectively $4\pi \sin^2 \rho$ and $\pi(2\rho - \sin 2\rho)$; in $\mathbf{H}^2$ they are respectively $4\pi \sinh^2 \rho$ and $-\pi(2\rho - \sinh 2\rho)$.

Apparently, sinh—the imaginary counterpart of sin, expressed

[3]And also to Franz Taurinus (1794-1874), who found it in 1825-1826 as ‘logarithmic-spherical’ geometry.

as $\sinh x = \sin ix/i$—replaces sin in the hyperbolic world: thus Lambert's discovery of hyperbolic trigonometry is apropos!

The situation generalises to surfaces of *any* constant curvature K. In most of the cases above, sin in the spherical formulae is replaced by

$$\sin_K(x) = \frac{\sin(\sqrt{K}x)}{\sqrt{K}}.$$

(It does not matter which square root we take, because $\sin(-z) = -\sin(z)$. $\cos_K(x)$ is defined as the derivative of $\sin_K(x)$.)

The general expression for the volume of a sphere is

$$\frac{1}{K}\pi(2\rho - \sin_K 2\rho).$$

Even Euclidean geometry ($K = 0$) joins in the fun, arising from a well-known limit:[4]

$$\sin_0 x = \lim_{K\to 0} \frac{\sin(\sqrt{K}x)}{\sqrt{K}} = x.$$

As the curvature of a sphere of radius r is r^{-2}, perhaps Euclidean geometry appears on a sphere of infinite radius! (And notably, $i^{-2} = -1$.)

One more example: the cosine rules in the three geometries, with sides a, b and c opposite angles α, β and γ. As above, we list them in the order: spherical, Euclidean, hyperbolic, with respective curvatures $+1, 0, -1$. (The Euclidean version is obtainable as a

[4] The volume of a Euclidean sphere likewise requires taking a limit; the familiar $\frac{4}{3}\pi\rho^3$ falls out of the cubic term in the Taylor series of sin.

limit.) The trigonometric functions applied to the *sides* change with the curvature, but those applied to the *angles* remain circular.

Spherical: $\cos a = \cos b \cos c + \sin b \sin c \cos \alpha ,$

Euclidean: $a^2 = b^2 + c^2 - 2bc \cos \alpha ,$

Hyperbolic: $\cosh a = \cosh b \cosh c - \sinh b \sinh c \cos \alpha .$

In spherical and hyperbolic geometries, there is an absolute unit of length, and thus similar triangles are congruent: AAA is now an actual congruence law![5] So it is not shocking that dual cosine rules, allowing one to solve for a side in terms of the angles, exist here. First we give the spherical form, then the hyperbolic.

Spherical: $\cos \alpha = -\cos \beta \cos \gamma + \sin \beta \sin \gamma \cos a ,$

Hyperbolic: $\cos \alpha = -\cos \beta \cos \gamma + \sin \beta \sin \gamma \cosh a .$

It is certainly convincing that hyperbolic geometry appears on imaginary spheres!

But how could we visualise that?

The Minkowski hyperboloid

With some relativistic trickery!

Spherical geometry takes place on the unit sphere

$$z^2 + x^2 + y^2 = 1$$

[5] In general, three pieces of information are always required to narrow a triangle down to finitely many possibilities (we cannot say *determine*, because of the ambiguous case of Euclidean SSA). The issue in Euclidean space is that because $\alpha + \beta + \gamma = \pi$ *always*, three angles are not actually three pieces of information, but are in fact only two. This is not a problem in spherical or hyperbolic geometry.

and one may naturally take an origin at the north pole, (0, 0, 1).

Euclidean geometry may naturally be considered to take place on the plane

$$z^2 = 1$$

with the origin again at (0, 0, 1).

So, the unit two-sheeted hyperboloid

$$z^2 - x^2 - y^2 = 1$$

should be a natural form for hyperbolic geometry. To preserve connectedness, we will only use the sheet with $z > 0$.[6]

And it is so, if we use the *Minkowski* notion of distance rather than the Euclidean one![7]

The x and y directions are *spacelike*: they have a positive contribution to the inner product. But the z direction is *timelike*, and has a negative contribution. Explicitly, the inner product is

$$(u_x, u_y, u_z) \cdot (v_x, v_y, v_z) = u_x v_x + u_y v_y - u_z v_z$$

The distance between two points is

$$d(u, v) = \operatorname{arcosh}(-u \cdot v)$$

[6] Resembling how we use the northern hemisphere alone to avoid the problem of antipodal points in elliptic geometry. (Though to get a projective plane we must additionally identify antipodal points on the equator $z = 0$.)

[7] Such a *pseudo-Euclidean* space is not a metric space: there are null vectors with zero length, forming a cone that physically is the *light cone* of the origin. Yet many things work in it nonetheless, such that we are moved to call the resulting notion of distance a *pseudo-metric*. The hyperboloid model within Minkowski space is a subset where distances are actually positive.

resembling a well-known formula in spherical geometry:

$$d(u, v) = \arccos (u \cdot v).$$

Our dream comes true: geodesics on the hyperboloid *are* straight lines in hyperbolic space. They take the role of great circles on the sphere.

As two orthogonal unit vectors u and v on the sphere trace out a great circle $u \sin\theta + v \cos\theta$, so an orthogonal spacelike unit vector u with $u \cdot u = 1$ and timelike unit vector v with $v \cdot v = -1$ trace out a hyperbolic geodesic $u \sinh\theta + v \cosh\theta$.

The familiar models of the hyperbolic plane are projections of this hyperboloid, corresponding to the map projections we use for the sphere!

- To get the Poincaré disc model from the Minkowski hyperboloid, we look at the hyperboloid from $(0, 0, -1)$; this model is analogous to stereographic projection of a sphere. See figure 1.
- The Beltrami–Klein model appears looking at the hyperboloid from $(0, 0, 0)$; it is analogous to gnomonic projection of a sphere.[8]

[8]These models—Beltrami–Klein for $\mathbf{H}^2$ and gnomonic for $\mathbf{S}^2$—are especially important because they map straight lines to straight lines. Hence, they are what you would actually see if you were magically transported to universes with such geometries. There is a strange irony here: because parallel lines diverge in hyperbolic space but converge in elliptic space, hyperbolic space looks finite to an internal observer used to Euclidean depth perception, but elliptic space looks infinite! An object at the antipodal point is seen from every direction at distance 2π, creating an impressive view.

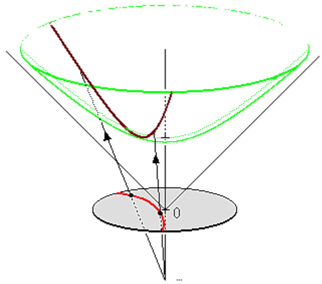

Figure 1. Projection of the Minkowski hyperboloid to the Poincaré disc model. One geodesic and its projection is shown. By Wikipedia user Selfstudier.

- Looking at the hyperboloid from $(0, 0, -\infty)$ gives the Gans model; it is analogous to orthographic projection of a sphere.[9]

Three geometries, three conics?

There remain differences between the three geometries. Most obviously, the sphere is bounded, but the Euclidean and hyperbolic planes are not. Yet even in their differences they show analogies!

Consider what one must add to let all lines intersect. Elliptic geometry needs nothing added. Euclidean geometry needs a line at

[9]For more such correspondences, see the video of Zeno Rogue (Eryk Kopczyński): `https://www.youtube.com/watch?v=H7NKhKTjHVE`. Check out his game HyperRogue (`https://roguetemple.com/z/hyper/`) as well: the properties of $\mathbf{H}^2$ are deeply used in the gameplay! It is also possible to play in $\mathbf{R}^2$ or $\mathbf{S}^2$, and in any of the eight *Thurston geometries* in three dimensions: $\mathbf{S}^3$, $\mathbf{E}^3$, $\mathbf{H}^3$, $\mathbf{S}^2 \times \mathbf{R}$ (surface of a four-dimensional spherinder), $\mathbf{H}^2 \times \mathbf{R}$, and three more exotic ones.

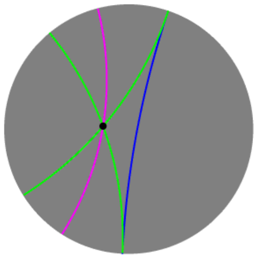

Figure 2. The green lines are limit parallel to the blue line. The magenta line is ultraparallel to the blue line. By Wikipedia user Tosha.

infinity (familiar from projective geometry). Hyperbolic geometry needs not only the *ideal* points on the boundary of the Poincaré disc (where *limit parallel* lines meet – they get asymptotically closer to each other, within any finite distance, but never touch), but also the *ultra-ideal* points outside it (where *ultraparallel* lines meet; they always keep a minimum positive distance from each other). The locations of ideal and ultra-ideal points are clear on the Beltrami–Klein model, where geodesics are drawn as Euclidean lines.

Now think of the conics that the geometries are named after! (Euclidean geometry is occasionally called *parabolic*.)

Consider circles in each projectively completed geometry. In elliptic geometry there are only circles: in Euclidean geometry lines are degenerate circles with infinite radius; but in hyperbolic geometry, a circle passes through other stages first, going through *horocycles* (limit circles) and *hypercycles* (equidistant curves, i.e. the locus of all points equidistant to a given line) before it becomes a line. A horocycle is simultaneously a circle whose centre has receded to infinity, and also an equidistant with the guiding line at infinity. See figure 2.

On the Minkowski hyperboloid, these may be seen as intersections of the hyperboloid with a plane. If the plane is tilted from the horizontal at an angle less than $\frac{\pi}{4}$ (i.e. its normal is timelike), the intersection is a circle; if equal to $\frac{\pi}{4}$ (a null or lightlike normal), a horocycle; if greater (a spacelike normal), a hypercycle. Their projections on the Poincaré disc are respectively circles not touching, tangent to, and secant to the boundary.

Another analogy shall be left for consideration. In elliptic geometry translations and rotations are one; in Euclidean geometry translations are limit rotations around infinity. What happens in hyperbolic geometry?

Three geometries, three complex numbers?

Finally, a lesser-known triad.

We all know the complex numbers: they are the real numbers, plus a square root of -1, which we call i.

$$i^2 = -1.$$

But there are two other lesser-known systems of two-dimensional numbers. One is the *dual numbers*: the reals, plus a nonzero square root of 0 called ϵ:

$$\epsilon^2 = 0.$$

The other is the *split-complex numbers* (or *double numbers*), where we add a nonreal square root of 1, called j:

$$j^2 = +1.$$

Any two-dimensional real unital (i.e. having a multiplicative identity) algebra[10] is isomorphic to one of these three as a ring.

Mimicking the derivation of Euler's formula recovers the generalised sines and cosines from the three geometries:

$$\begin{aligned} e^{ix} &= \cos x + i \sin x, \\ e^{\epsilon x} &= 1 + \epsilon x, \\ e^{jx} &= \cosh x + j \sinh x. \end{aligned}$$

What's going on here? How far does complex analysis generalise? And how do dual numbers relate to infinitesimals?

References and further reading

For more on the Minkowski hyperboloid model:

- Reynolds, W. F. (1993). *Hyperbolic Geometry on a Hyperboloid*. The American Mathematical Monthly, *100*(5), 442-455.

For more on Lambert's work:

- Papadopoulos, A. & Théret, G. (2014). *Hyperbolic geometry in the work of Johann Heinrich Lambert*. Ganita Bharati (Indian Mathematics):

[10]Traditionally, such structures in any dimension are called *hypercomplex numbers*. Interesting examples arise from the Cayley–Dickson construction and as Clifford algebras.

Journal of the Indian Society for History of Mathematics, *36*(2), 129-155. https://hal.science/hal-01123965/document

For more on the analogies between the three geometries:

- *Similarities between non-Euclidean geometries*, Math Stack Exchange (https://math.stackexchange.com/questions/2768462)
- Andreas Bloch et al., *A Universal Model for Hyperbolic, Euclidean and Spherical Geometries* (https://andbloch.github.io/K-Stereographic-Model/).

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- Weeks, J. R. (2002). *Real-Time Rendering in Curved Spaces*. IEEE Computer Graphics and Applications, *22*(6), 90-99. https://page.math.tu-berlin.de/\textasciitildesullivan/L/03W/Vis/curved-sp.pdf
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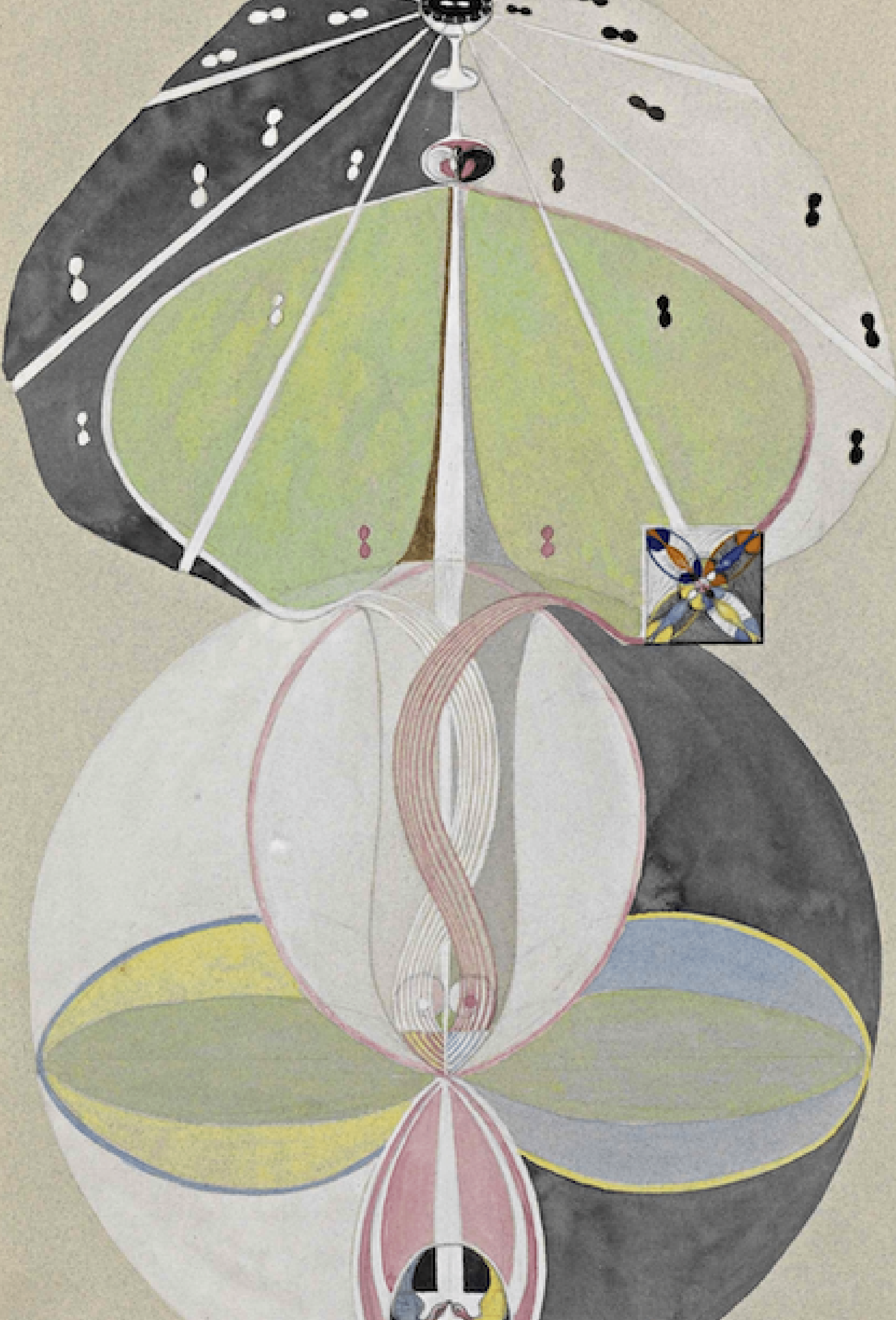
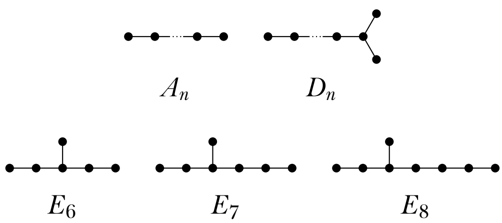


ADE classification but not Lie algebras

Yourong Zang

John McKay's observation

It is a well-known result that simple Lie algebras can be classified by *Dynkin diagrams*. Those corresponding to $\mathfrak{sl}_{n+1}$, $\mathfrak{so}_{2n}$ and exceptional Lie algebras $\mathfrak{e}_i$ for $i = 6, 7, 8$ are *simply-laced* (n corresponds to the number of nodes).



Yet it is sometimes lesser known that these graphs of type A, D, E naturally correspond to finite subgroups of $SU_2(\mathbb{C})$. The ancient Greek mathematicians already knew there were only five regular polyhedra: tetrahedron, cube, octahedron, dodecahedron, and icosahedron. One might consider groups of rotations on these platonic solids as finite subgroups of SO_3, which can be pulled back to finite subgroups of SU_2 via a homomorphism $SU_2 \to SO_3$ with kernel $\{\pm I\}$. Thus it is not unreasonable for the reader to assume there exists some classification of the finite subgroups of SO_3 or SU_2. However, it was not until the late 19th century that they were completely classified. Felix Klein studied these finite groups extensively and was known to be the first person who classified them in [1].

Theorem (Klein)**.** *A finite subgroup of $SU_2(\mathbb{C})$ is conjugate to one of the following types:*

1. *Type A_{n-1}: a cyclic group of order n,*
2. *Type D_{n+2}: a binary dihedral group BD_n,*
3. *Type E_6: the binary tetrahedral group BT,*
4. *Type E_7: the binary octahedral group BO,*
5. *Type E_8: the binary icosahedral subgroup BI.*

We present their generators in the following table.

Type	Generators	z
A_{n-1}	$\begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix}$	$e^{2\pi i/n}$
D_{n+2}	$\begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$	$e^{\pi i/n}$
E_6	$\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\frac{1}{\sqrt{2}}\begin{bmatrix} z & z^3 \\ z & z^{-1} \end{bmatrix}$	$e^{\pi i/4}$
E_7	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\frac{1}{\sqrt{2}}\begin{bmatrix} z & z^3 \\ z & z^{-1} \end{bmatrix}$, $\begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix}$	$e^{\pi i/4}$
E_8	$\begin{bmatrix} z^3 & 0 \\ 0 & z^2 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\frac{1}{\sqrt{5}}\begin{bmatrix} -z+z^4 & z^2-z^3 \\ z^2-z^3 & z-z^4 \end{bmatrix}$	$e^{2\pi i/5}$

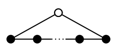
Table 1. Generators of finite subgroups of $SU_2(\mathbb{C})$

.

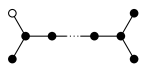
Nonetheless, Klein's classification is unrelated to the aforementioned Dynkin diagrams of type A, D, E. The correlation was much later observed by John McKay in [2]. Given a finite subgroup G of SU_2, we may study its *representations*, i.e., homomorphisms from G to the group of isomorphisms $GL(V)$ on some vector space. If you have taken a first course in group representation theory, you probably know any representation of G decomposes into a direct sum of copies of *irreducible* representations, which are representations not "containing" any smaller nonzero representations. I'm sure in this course you also learned that there are only finitely many irreducible representations $V_1, \ldots, V_n$ of G up to isomorphisms (we here include the trivial representation). There is also an obvious representation $V_0 = \mathbb{C}^2$ of G given by $\rho : G \hookrightarrow GL(V_0)$ sending a matrix in G to itself in $GL(V_0)$. Decomposing the tensor product of V_0 and each V_i, we get a sum $V_0 \otimes V_i = \oplus_j V_j^{\oplus r_{ij}}$. McKay defined a graph using these decompositions and made the following observation:

Definition. The **McKay graph** $\Gamma(G)$ of G is a graph in which each node corresponds to an irreducible representation of G, and nodes i, j are connected by r_{ij} edges.

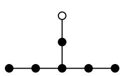
Theorem (McKay correspondence). *The map $G \mapsto \Gamma(G)$ is a bijection between (conjugate classes of) finite subgroups of SU_2 and the* extended Dynkin diagrams *respecting their types*



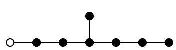
$\tilde{A}_n$



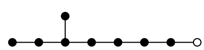
$\tilde{D}_n$



$\tilde{E}_6$



$\tilde{E}_7$



$\tilde{E}_8$

Let us verify this for $G = C_n$ the cyclic group of order n corresponding to the type A_{n-1}. This group is generated by the diagonal matrix $\mathrm{diag}(\xi, \xi^{-1})$ where $\xi = e^{2\pi i/n}$. It has irreducible representations $\rho_m : \xi^i \mapsto \xi^{mi}$ for $m = 1, \ldots, n$ and clearly the representation

V_0 is a direct sum $\rho_1 \oplus \rho_{n-1}$. You may use the fact that $\rho_i \otimes \rho_j = \rho_{i+j}$, and that tensor product distributes over direct sums to show that $\Gamma(G)$ indeed has type A_{n-1}. I encourage readers who are familiar with character theory to verify the above for other subgroups. One can in fact prove the correspondence using pure algebra and character theory. But this might be a bit tedious for geometers. In the rest of this article, we will go over a beautiful geometric theory that leads to the same result.

Resolving Klein singularities

Recall it was Klein who classified finite subgroups of SU_2. Here is what I did not tell you: Klein actually did much more than this. He associated each type of finite group with a unique surface in $\mathbb{C}^2$. Still, we start with a finite subgroup G in SU_2. The matrices in G act on $\mathbb{C}^2$ and I'm sure you have done a lot of these in your first linear algebra course. If we consider the algebra of polynomials in two variables $\mathbb{C}[x, y]$ as the algebra of functions on $\mathbb{C}^2$, there is an action of G on $\mathbb{C}[x, y]$: let $g = \left[\begin{smallmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{smallmatrix}\right] \in G$, we define $(g \cdot f)(x, y)$ to be $(f \circ g^{-1})(x, y) = f(g_{22}x - g_{12}y, -g_{21}x + g_{11}y)$. We denote by $\mathbb{C}[x, y]^G$ the set of invariant polynomials. You can go ahead and show it is an algebra. You probably wonder if there is an explicit description of it. Let's take $G = C_n$ the cyclic group of order n as an example. Invariant polynomials are sums of invariant monomials in x, y. A monomial $x^a y^b$ is invariant under the generator $\operatorname{diag}(\xi, \xi^{-1})$ (still $\xi = e^{2\pi i/n}$) if and only if $\xi^{b-a} = 1$, meaning $b - a \equiv 0 \pmod{n}$. Thus, $\mathbb{C}[x, y]^G$ is the algebra $\mathbb{C}[x^n, y^n, xy]$. If we consider the map $\mathbb{C}[x, y, z] \to \mathbb{C}[x, y]^G$ defined by $x \mapsto x^n$, $y \mapsto y^n$ and $z \mapsto xy$, we get an isomorphism $\mathbb{C}[x, y]^G \cong \mathbb{C}[x, y, z]/(z^n - xy)$.

Regardless of your knowledge of naive algebraic geometry, you can intuitively see that the quotient $\mathbb{C}[x, y, z]/(xy - z^n)$ is the algebra of polynomial functions on the surface in $\mathbb{C}^3$ defined by $xy - z^n = 0$. In fact, Klein showed that

Proposition. *The finite subgroup of SU_2 of type*

1. *A_{n-1} corresponds to the surface $xy - z^n = 0$,*
2. *D_{n+2} corresponds to the surface $x^2 + z(y^2 - z^n) = 0$,*
3. *E_6 corresponds to the surface $x^2 + y^3 + z^4 = 0$,*
4. *E_7 corresponds to the surface $x^2 + y^3 + yz^3 = 0$,*
5. *E_8 corresponds to the surface $x^2 + y^3 + z^5 = 0$.*

We call these surfaces Klein singularities*.*

From now we denote by $\mathbb{C}^2/G$ the Klein singularity associated with G (if the reader is familiar with Mumford's geometric invariant theory, then they probably realize we are just constructing an affine quotient for the action of G on $\mathbb{C}^2$). The reader might be curious about the term 'singularity'. Indeed, the point $(0, 0, 0)$ in each surface above is a *singularity*. Vaguely, we can't do calculus as usual at this point on the surface. To resolve these singularities, we need some nice (in our case smooth complex) surfaces that almost look like $\mathbb{C}^2/G$.

Definition. A **resolution** of the singularity $0 = (0, 0, 0)$ is a smooth complex surface Y equipped with a proper morphism $Y \to \mathbb{C}^2/G$ such that the restriction $\pi : \pi^{-1}(\mathbb{C}^2/G \backslash \{0\}) \to \mathbb{C}^2/G \backslash \{0\}$ is an isomorphism.

If the terms 'proper morphism' and 'isomorphism' sound daunting to you, you can intuitively picture them as surjections and bijections. The preimage $\pi^{-1}(0)$ of 0 is called the *exceptional divisor*. We are mainly interested in a resolution that is *minimal*, i.e., a resolution $\pi : \widetilde{\mathbb{C}^2/G} \to \mathbb{C}^2/G$ such that every resolution of $\mathbb{C}^2/G$ factors through π. If a minimal resolution of $\mathbb{C}^2/G$ exists, the exceptional divisor is a connected union of components E_i isomorphic to $\mathbb{P}^1$ and self-intersect with an intersection number of -2. Patrick du Val proved the following classification of singularities in [4].

Theorem (Du Val). *Suppose there is a minimal resolution* $\widetilde{\mathbb{C}^2/G}$ *of* $\mathbb{C}^2/G$ *with an exceptional divisor* $\cup_i E_i$*. Construct the graph* $\Gamma'(G)$ *using the following rules: add a node* i *for each component* E_i *in the exceptional divisor and add an edge between nodes* i, j *if* E_i *intersects* E_j*. Then* $\Gamma'(G)$ *is a regular Dynkin diagram of the same type as* G*.*

Note here we use regular Dynkin diagrams. Those in the previous McKay correspondence are extensions of them. They can be made consistent if we only use nontrivial irreducible representations of G to build the McKay graphs. The theorem can be best illustrated by figure 1 from [3].

The surface on the left is $x^2+z(y^2-z^2) = 0$ of type D_4 (draw it on your computer!), and the thing on the right is a minimal resolution. Note that the singular point in the center of the surface is resolved to four small circles on the right (which are just $\mathbb{P}^1$). Their intersections precisely produce the Dynkin diagram of type D_4

You might find the previous few paragraphs unclear; there seem

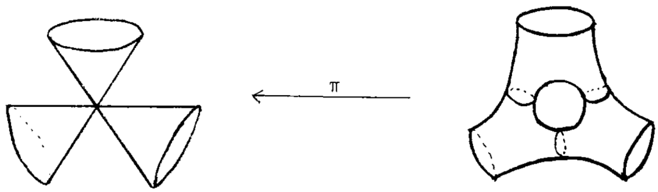


Figure 1. The surface $x^2 + z(y^2 - z^2) = 0$ on the left and its minimal resolution on the right.

to be too many ideas and constructions involved. So I will demonstrate a simple case using a fairly approachable method of resolution. If you have some experience in algebraic geometry or complex geometry, you probably would've guessed it by now. We will *blow-up* the singularity. Namely, consider the space $\tilde{\mathbb{C}^3} \subseteq \mathbb{C}^3 \times \mathbb{P}^2$ defined by

$$\tilde{\mathbb{C}^3} = \{((x, y, z), [z_0 : z_1 : z_2]) : (x, y, z) \in [z_0 : z_1 : z_2]\}$$

where a point $[z_0 : z_1 : z_2] \in \mathbb{P}^2$ is considered as a line in $\mathbb{C}^3$. Let $p : \tilde{\mathbb{C}^3} \to \mathbb{C}^3$ be the projection to the first coordinate. Now for any closed surface $Y \subseteq \mathbb{C}^3$, we blow up Y at the origin by taking $\tilde{Y} = \overline{p^{-1}(Y \backslash 0)}$. Despite the tilde notation, blowing up at a point does not guarantee a resolution. But for Klein singularities, we can obtain a minimal resolution by successively blowing up at singular points!

We will end this article by computing the blow up of the simplest surface Y of type A_1 and constructing the corresponding Dynkin diagram. Recall that the algebra of functions on Y is given by $\mathbb{C}[x, y, z]/(xy - z^2)$. Blowing the surface up at zero we obtain the space $U \cup V$ where

$$\begin{aligned} U &:= \{((x, y, z), [x : y : z]) \in \mathbb{C}^3 \backslash \{0\} \times \mathbb{P}^2 : xy - z^n = 0\}, \\ V &:= \{((0, 0, 0), [a : b : c]) : ab - c^2 = 0\}. \end{aligned}$$

This space can be covered by smooth surfaces, so we don't need to blow up again (but we need to blow up again even for A_2). The exceptional divisor on the right is a conic in $\mathbb{P}^2$, hence isomorphic to $\mathbb{P}^1$. Therefore, the Dynkin diagram of Y is given by only one node. Too simple? This is the furthest we can reach without better tools.

The theory of resolutions is very rich. There are so many ways one could construct a resolution. To name a few: iterated blow ups, Hilbert schemes, quiver varieties, etc. Each of them is an extremely interesting and deep object to study. If you find the simple example above interesting and unsatisfying, you can read about blow ups of $xy - z^n = 0$ and see if you can derive the corresponding Dynkin diagrams.

We witnessed an ADE classification and went through some simple examples and computations. But this is only *one* example of ADE classifications. There are many others that are open for you to explore. The techniques we used also indicate a connection between representation theory and geometry. A modern generalization of the McKay correspondence is the study of derived categories of quasicoherent sheaves on the quotient $\mathbb{C}^2/G$ and quotients in higher dimensions. Classification is only a starting point. The true nature of simple things lies in sophisticated but elegant theories.

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[1]. Felix Klein, *Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade*, Teubner, Leipszig, 1884.

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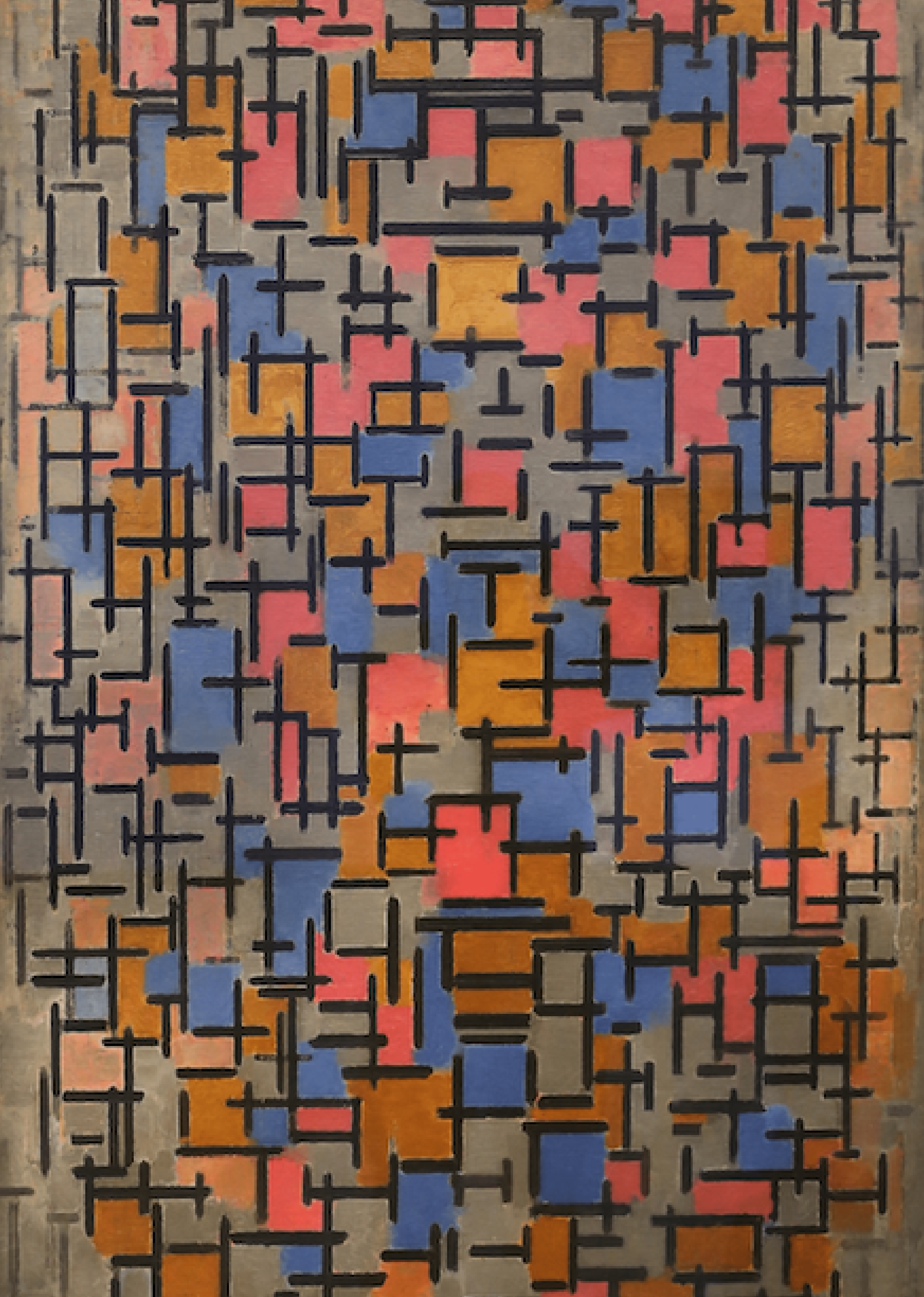


The Ising Model and Phase Transitions

Ghazi Zein

Introduction

Many proofs are adapted from Friedli and Velenik's book on Statistical Mechanics.

In 1895, Curie studied the phenomenon whereby a magnet shifts from ferromagnetism (ability to spontaneously magnetise) to paramagnetism (magnetises only in the presence of an external field) once heated above the Curie temperature. The shift is sharp, not gradual. Later on, to explain this phenomenon, Lenz proposed a first model, which was then further developed and refined by his student Ising in his 1925 paper into what is now known as the Ising model.

What is a phase transition? There are two radically different ways of defining a *first order phase transition*: either by viewing it as the non-differentiability of a **pressure function**, or by viewing it as the non-unicity of a **Gibbs state**. In this article we shall propose two seemingly different definitions for phase transition, and prove that they are totally equivalent.

The Ising Model

Definitions

We start with the finite volume Ising model with free boundary conditions. We will restrict ourselves to $\Lambda \Subset \mathbb{Z}^d$ (Λ is a *finite* subset of $\mathbb{Z}^d$). d here is the dimension of interest.

Definition. Configurations of the Ising model on the infinite lattice are given by $\Omega := \{+1, -1\}^{\mathbb{Z}^d}$.
A configuration of the Ising model in Λ with boundary condition η is an element of

$$\Omega_\Lambda^\eta := \{\omega \in \Omega \mid \omega_i = \eta_i, \forall i \notin \Lambda\}.$$

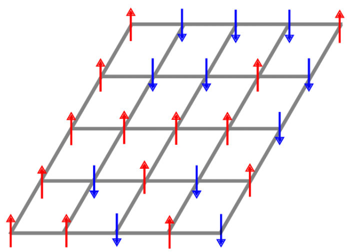


Figure 1. A possible configuration.

Definition. To each configuration, we associate its energy, given by the Hamiltonian

$$\mathscr{H}_{\Lambda;\beta,h}(\omega) := -\beta \sum_{(i,j)\text{nearest neighbours}} \sigma_i(\omega)\sigma_j(\omega) - h \sum_{i \in \Lambda} \sigma_i(\omega)$$

where $\sigma_i \in \{+1, -1\}$ is the spin at node i, $\beta \in \mathbb{R}_{\geq 0}$ is the inverse temperature, and $h \in \mathbb{R}$ is the external field strength.

Definition. The Gibbs distribution in Λ with boundary condition η, at parameters β and h, for all $\omega \in \Omega_\Lambda^\eta$, is given by

$$\mu_{\Lambda;\beta,h}^\eta(\omega) := \frac{1}{\mathbf{Z}_{\Lambda;\beta,h}^\eta} \exp(-\mathscr{H}_{\Lambda;\beta,h}(\omega)).$$

The normalisation constant

$$\mathbf{Z}_{\Lambda;\beta,h}^\eta := \sum_{\omega\in\Omega_\Lambda^\eta} \exp(-\mathscr{H}_{\Lambda;\beta,h}(\omega))$$

is called the **partition function** with boundary condition η.

Magnetisation

We now build up the first definition of a *phase transition*.

Definition. In the thermodynamic limit, the **pressure** is defined by

$$\psi(\beta,h) := \lim_{\Lambda\Uparrow\mathbb{Z}^d} \frac{1}{|\Lambda|} \log \mathbf{Z}_{\Lambda;\beta,h}^\#$$

We can show that the pressure is well defined, independent of the sequence $\Lambda \Uparrow \mathbb{Z}^d$ and of the type of boundary condition.

To understand why we named this function pressure, we can interpret the $+1$ spins as particles and the -1 spins as vacancies, hence creating a 'pressure'.

Definition. The **magnetisation density** is defined by

$$m_\Lambda^\#(\beta,h) := \langle m_\Lambda \rangle_{\Lambda;\beta,h}^\#,$$

and the **average magnetisation density** is defined by

$$m(\beta,h) := \lim_{\Lambda\Uparrow\mathbb{Z}^d} m_\Lambda^\#(\beta,h)$$

where $m_\Lambda = \frac{1}{|\Lambda|}\sum_{i\in\Lambda}\sigma_i$ is the magnetization density and $\langle\cdot\rangle$ is the expectation with respect to the Gibbs measure.

There is a strong link between **pressure** and **magnetisation.**

Theorem 1. *For all $\beta \geq 0$ and $h \in \mathbb{R}$:*

$$\frac{\partial\psi}{\partial h^+}(\beta, h) = m^+(\beta, h), \;\; \frac{\partial\psi}{\partial h^-}(\beta, h) = m^-(\beta, h).$$

We see that if there is no phase transition, then as $h \uparrow 0$ or $h \downarrow 0$, the derivatives coincide:

$$m(\beta, h) = \frac{\partial\psi}{\partial h}(\beta, h).$$

Intuitively, a shift from ferromagnetism to paramagnetism means the magnetisation graph becomes discontinuous at $h = 0$, i.e.

$$\lim_{h\downarrow 0} m(\beta, h) \neq \lim_{h\uparrow 0} m(\beta, h).$$

This is because you would obtain a 'residual' magnetic field, i.e. when there is no external field ($h = 0$), you have spontaneous magnetisation.

With all this in place, we are now ready for the first definition of phase transition.

Definition. The pressure ψ exhibits a **first-order phase transition** at (β, h) if $h \mapsto \psi(\beta, h)$ fails to be differentiable at that point.

Infinite-Volume Gibbs States

Although magnetization seems more intuitive, it proves to be more difficult to work with in practice. We can leverage the properties of

Gibbs states to build an easier and more comprehensive method of finding phase transitions.

We start off this section by defining a very useful tool, *local functions*.

Definition. A function $f : \Omega \to \mathbb{R}$ is *local* if there exists $\Delta \Subset \mathbb{Z}^d$ such that $f(\omega) = f(\omega')$ as soon as ω and ω' coincide on Δ. The support of f is defined as the smallest such set Δ.

It is also useful to know that local functions possess summation representations in terms of products of spins over subs-sets of their support. This is very useful for proofs because it enables us to start with smaller cases and then generalise to the entire function.

Now that we have defined local functions, we can also define a *Gibbs state*.

Definition (Gibbs state at (β, h))**.** Let $\Lambda_n \uparrow \mathbb{Z}^d$, $(\#_n)_{n\geq 1}$ be a sequence of boundary conditions. The sequence of Gibbs distributions $(\mu^{\#_n}_{\Lambda_n;\beta,h})_{n\geq 1}$ converge to the *Gibbs state* $\langle\cdot\rangle$ at (β, h) if and only if $\lim_{n\to\infty}\langle f\rangle^{\#_n}_{\Lambda_n;\beta,h} = \langle f\rangle$ for every local function f.

We can construct two translation invariant Gibbs states, namely $\langle\cdot\rangle^-_{\beta,h}$ and $\langle\cdot\rangle^+_{\beta,h}$, where any other Gibbs state is squeezed in between the two (they are lower and upper bounds respectively). This is important, as the following theorem can be proved:

Theorem 2. *There exists a unique Gibbs state at* $(\beta, h) \Leftrightarrow \langle\sigma_0\rangle^+_{\beta,h} = \langle\sigma_0\rangle^-_{\beta,h}$.

Sketch of proof. ($\Leftarrow$) We already have $\langle\sigma_0\rangle^+_{\beta,h} \geq \langle\sigma_0\rangle^-_{\beta,h}$. The other inequality can be shown using the Fortuin–Kasteleyn–Ginibre inequality.

($\Rightarrow$) This is straightforward. □

We are now ready for the second definition of phase transition.

Definition. If at least two distinct Gibbs states can be constructed for a pair (β, h), we say that there is a first-order phase transition at (β, h).

Characterisation of Uniqueness

Now we introduce a unifying theorem.

Theorem 3. *$h \mapsto \psi(\beta, h)$ is differentiable at h_0 if and only if there exists a unique Gibbs state at (β, h_0).*

Sketch of proof. From Theorem 1, $\frac{\partial \psi}{\partial h}(\beta, h)$ exists if and only if $m^+(\beta, h) = m^-(\beta, h)$. If we show that $m^+(\beta, h) = \langle \sigma_0 \rangle^+_{\beta,h}$ and $m^-(\beta, h) = \langle \sigma_0 \rangle^-_{\beta,h}$, then this theorem follows from Theorem 2.

We will now deal with the + case. The − case can be dealt with in a similar fashion. It can be shown that

$$\langle \sigma_0 \rangle^+_{\beta,h} = \langle m_{\Lambda_n} \rangle^+_{\beta,h} \le \langle m_{\Lambda_n} \rangle^+_{\Lambda_n;\beta,h} \le \langle \sigma_0 \rangle^+_{\mathrm{B}(k);\beta,h} + 2\frac{|\mathrm{B}(k)||\partial^{\mathrm{in}}\Lambda_n|}{\Lambda_n}$$

where $B(k) := \{-k, \ldots, k\}$, and we can then use the squeeze theorem for $n \to \infty$ to conclude. □

Outlook

So what have we demonstrated? We started off with some intuitive physical descriptions of a spin-lattice system and set up the first definition of phase transition with the tools we had available. We then leveraged powerful statistical tools to define Gibbs measures and Gibbs states, thereby creating a robust framework to work with our system: indeed, when we are dealing with vast quantities of particles (think Avogadro's number), we cannot account for individual particles anymore.

There is plenty more to talk about: the fact that no phase transition exists in 1D, that the 2D Ising model does indeed have a phase transition, higher dimensional models, new advancements using percolation theory (see Hugo Duminil-Copin, one of the latest Fields medal winners). I highly recommend exploring this rich subject!

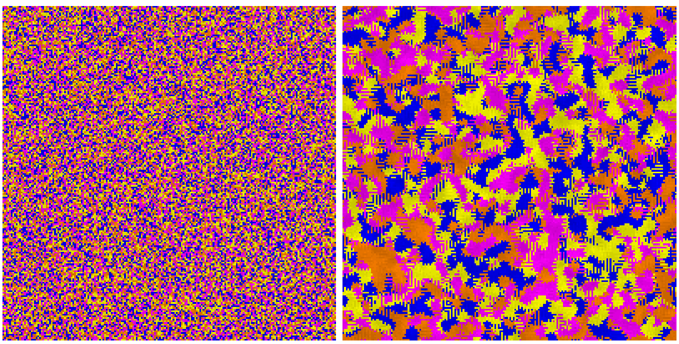
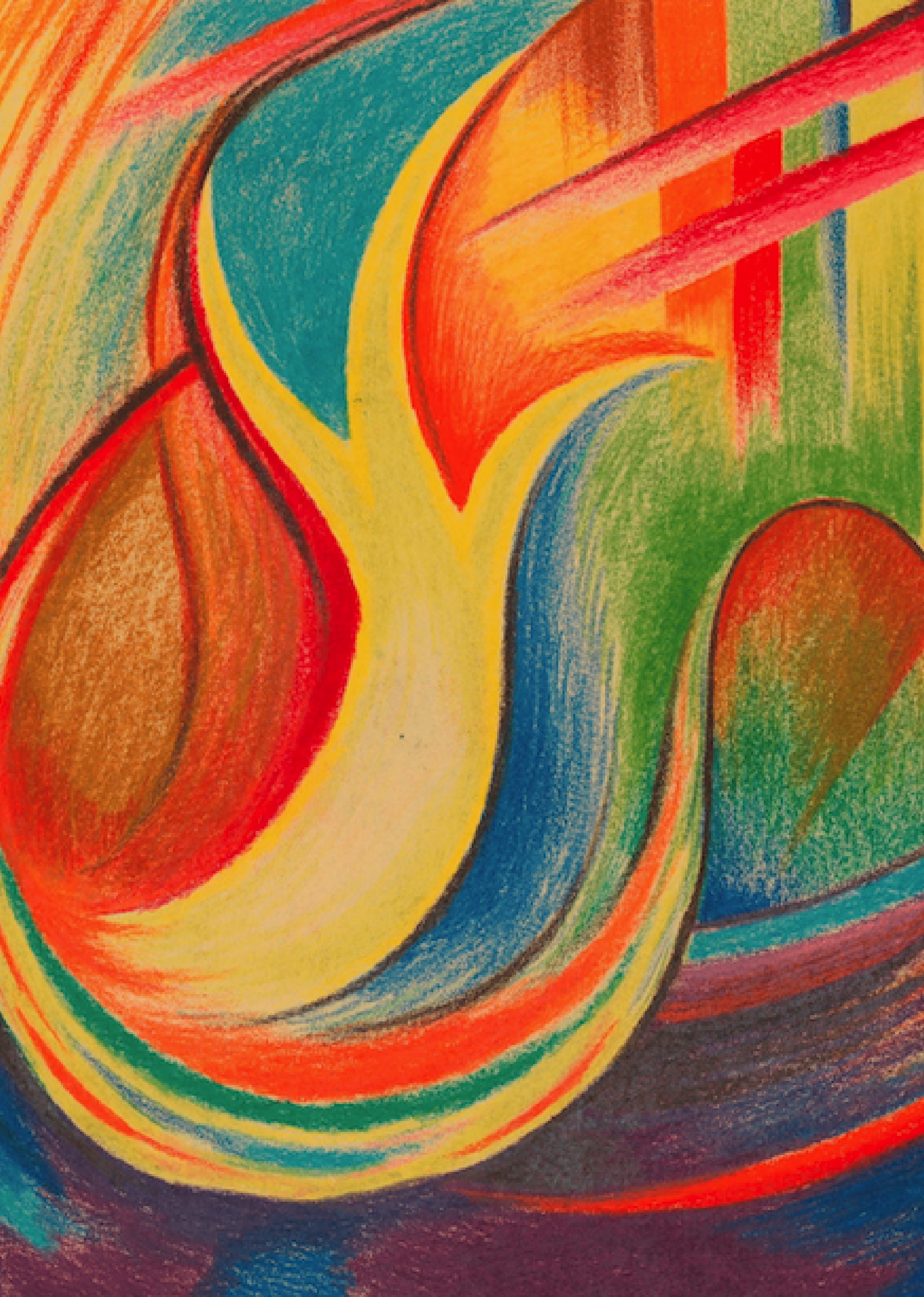


Figure 2. A preferential Ising model after 10 iterations.



Dictionary of Received Mathematical Ideas

Tian Long Lee

Because I do not hope to turn
Desiring this man's gift and that man's scope
I no longer strive to strive towards such things
—T.S. Eliot

Ours is a curious subject. It has curious conventions. For instance, the greenhorn mathematician feels a tinge of disappointment when, on opening a mathematical paper, they see that it does not consist solely of one monumental theorem and its gargantuan proof, but is instead subdivided into several smaller statements—definitions, theorems, examples, etc. But the veteran knows that this subdivision is necessary—the mind of a mathematician is necessarily bounded, and we must understand theories piecewise if we are to understand them at all. Those smaller statements are the many necessary bricks upon which the grand mathematical results—the crowning domes—are to be placed. Below is a list of the most common such statements, in no particular order, with some elucidating observations and helpful discussions for the mathematician, expert or novice alike.

Definition. A precise explanation of the mathematical meaning of a word, phrase, or symbol. Typically these are the first things the reader sees in an article or set of lecture notes. Every field of mathematics is based on good and correct definitions, and to formulate such definitions is no mean feat. It might take decades or centuries for mathematicians to come up with a definition that can be expressed in a single line.

Notation. From the mathematical laity one occasionally hears that mathematics is a mere play of symbols and funny squiggles. Be that as it may, a good author will always establish at the beginning of their work the notation that they will employ consistently throughout their work. Some advice: use triple subscripts because you will or you must, not because you can; avoid the quantifiers $\forall$ (for all), $\exists$ (there exists) when writing discursively; and never give multiple meanings to the same variable. The least a mathematician can do is clean up after themself.

Lemma. A mathematical statement that is used in proving other statements. Ideally, it should be true. For authors with a more literary (or pretentious) bent, lemmata is the correct Greek plural. It is a rather sad fact that many of the most important mathematical results are branded with the lowly and utilitarian-sounding designation of 'lemma'. Urysohn's Lemma and Bézout's Lemma are fundamental in the respective fields of general topology and ring theory. And without the Snake Lemma, where would our poor homology groups go? Some lemmas, like Ito's, have birthed whole branches of mathematics, but alas! there is no changing their name now...

Example. Hilbert was right when he insisted to start always with the simplest examples. Examples serve to instantiate definitions and theorems. Generally they do not add to the 'plot' of a mathematical treatise—they are not logically necessary—but should provide intuition for mathematical concepts. In fact, examples should come before the big theorems and definitions. We should not abstractly define an object like a ring and then go about finding examples that fit it; instead, our intuitions are better served by showing that matrices, integers, polynomials all share similar properties when we add and multiply them. They all have similar structures, and it is this structure we call a ring. We might do worse than to agree with Aristotle when he writes that the individual comes before the abstract—after all, the concrete examples are the reason that the abstract definitions exist! That is the natural path that mathematics has always taken, and it is the natural path that students learn about difficult mathematical concepts. Would that more educators knew that!

Non-Example. The dual of examples, non-examples are used to disprove statements which might at a first glance seem reasonable, but are in fact not true. In other places, non-examples are used to show that a certain assumption in a theorem is in fact necessary. All Riemann-integrable functions on the unit interval $[0, 1]$ are bounded, and just after the reader first encounters this, they are distraught to discover that the converse is not true: there do exist bounded functions on $[0, 1]$ which are not Riemann-integrable! This is demonstrated by a non-example.

Axiom. A mathematical statement that is assumed to be true, to serve as a premise or starting point for further reasoning and deductions. It is assumed to be true, but is it? Who cares—mathematicians will take it to be so. Remember that famous dictum: 'the Axiom of Choice is obviously true, the Well-Ordering theorem obviously false; and who can tell about Zorn's Lemma?'. Whatever the case may be, these controversial statements are given the designation of axiom, ensuring their philosophical weight.

Exercise. The bane of a student's academic life. Is there anything worse than trawling for hours through lecture notes or a textbook, toiling steadily on as the midnight oil burns, and coming across, as your eyes droop, the phrase 'the proof of this theorem is left as an exercise'? The phrase is often employed by intelligent lecturers wishing to avoid having to write up a tedious proof in LaTeX. That being said, there may be something to the idea that solving exercises by themself aids in the positive development of the student's mathematical abilities.

Proposition. A catch-all term for statements which are not important enough to be theorems, nor useful enough to be called lemmas. Often propositions collect properties of a certain mathematical object, or are semi-interesting standalone results, but not interesting enough to deserve of the label 'theorem'. Like lemmas, if you write one, it better be true.

Corollary. A true statement deduced from another result. The reader will often breathe a sigh of relief on reading the word 'Corollary': these tend to be short, simple, and satisfying deductions that follow quickly from a given theorem or proposition. This relief is sometimes confounded, however, when to the reader's chagrin they discover that the corollary requires more work than previously thought, and that it is not an immediate logical consequence of the result that precedes it.

Remark. An observation which brings attention to a certain issue that is not strictly relevant to the logical flow of the text, but to which the author may wish to refer later. Like examples, remarks may be used to offer motivation, by mentioning practical applications of a given concept, by bringing to the fore pitfalls that might ensnare the unsuspecting reader, or by justifying a seemingly circuitous route at a certain point of the text.

Comment. Usually less helpful and less pertinent than a remark, a comment serves to provide a commentary or contextual information about a mathematical idea, proof, or result. Sometimes, if the author is feeling bold, a comment may even include a joke or a 'fun' historical fact. Fact: Did you know that Galois died in a duel at the age of 20? Joke: What does the B in Benoît B Mandelbrot stand for? Answer: Benoît B Mandelbrot.

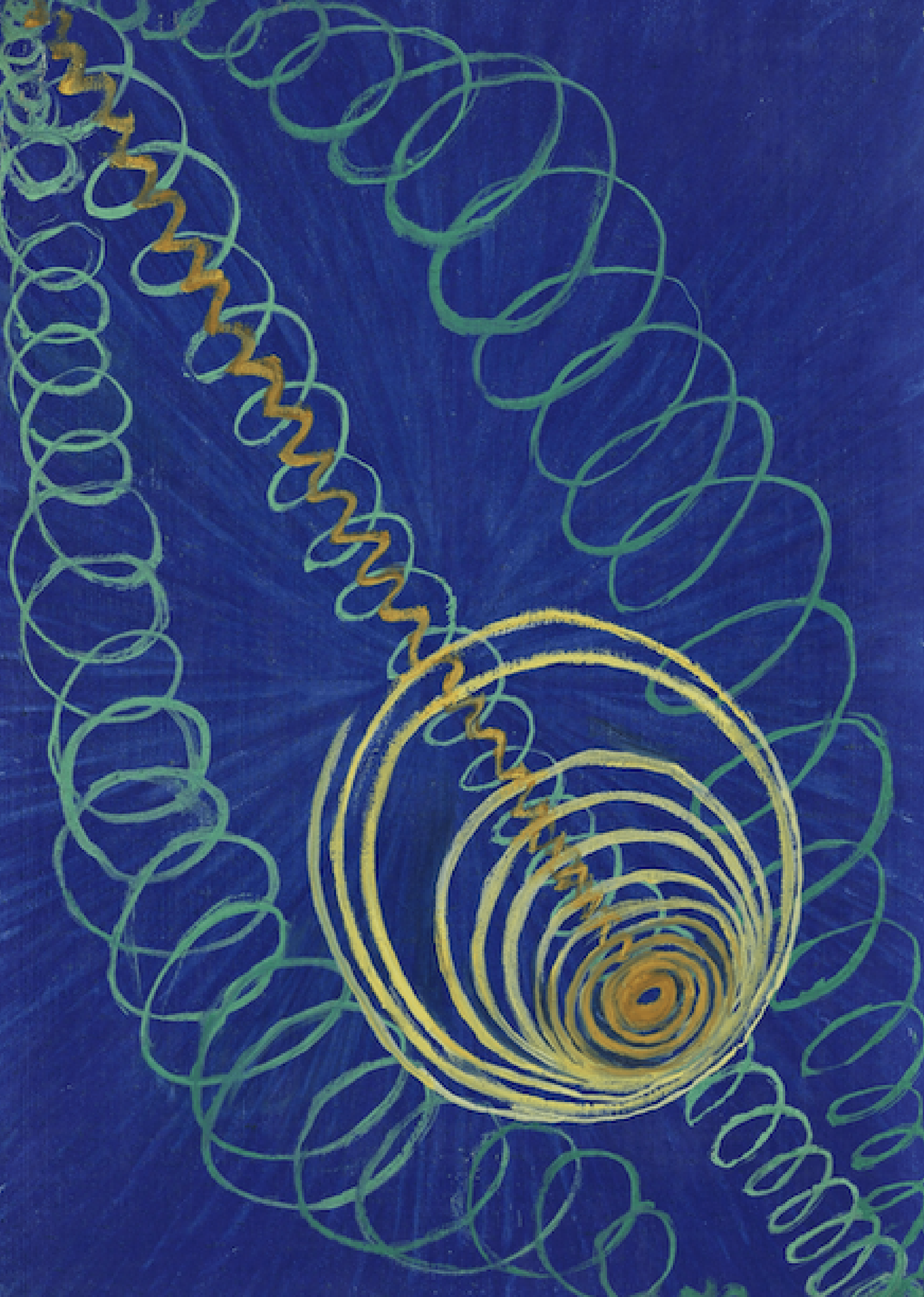
Conjecture: If the reader is lucky enough to be at the cutting edge of a mathematical field, they will in their readings occasionally come across the word 'conjecture'. A conjecture is an assertion that is thought to be true but that is yet to be proved. Of course, sometimes they turn out to be false. Some notable conjectures: the Riemann Hypothesis, the Twin Prime Conjecture, the Collatz Conjecture. When might these be solved? Next year? Next millennium? One feels a sigh of grandeur on reading or pronouncing the word 'conjecture'—one thinks of mountains yet to be climbed, or jungles unexplored. A great conjecture is like the moon that shows the darkness that it cannot dispel—or better, it serves as the sole lodestar dimly guiding the course of a whole branch of mathematics.

Proof. A proof is a mathematical demonstration of why a statement is true. Mathematics is a difficult subject. It is made all the more difficult by the requirement of having to prove every assertion that one makes. On the subject of proof I can do no better than to quote the great G. H. Hardy: In great proofs 'there is a very high degree of unexpectedness, combined with inevitability and economy. The arguments take so odd and surprising a form; the weapons used seem so childishly simple when compared with the far-reaching results; but there is no escape from the conclusions.' Depending on the author's aesthetic inclinations, proofs may end with either 'QED' or a square. □

Theorem. A theorem is an important statement that has been proved to be true. Like in the case of lemmas, some theorems are straightforwardly named: the Open Mapping Theorem (if a bounded linear operator between Banach spaces is surjective then it is an open mapping) or the Prime Ideal Theorem (every proper ideal is contained in some prime ideal). Others are rather more suggestive: the Hairy Ball Theorem, asserting that there does not exist a non-vanishing continuous tangent vector field on the n-sphere for even integers n—it has the interpretation that given a sphere with hairs all over it, it is impossible to comb the hairs such that all the hairs lay flat: at least one hair must be sticking straight up.

Some great theorems are named after great eponyms—Tychonoff's, Euclid's, Dirichlet's. These mathematicians strove and shed blood, sweat, and tears so that we can say with certainty that these theorems are true, and for all their toil they are rewarded by having their names writ into eternity. But some mathematicians get it lucky: Fermat's Last Theorem was manifestly not proven by Pierre de Fermat; the Heine-Borel Theorem was proved only by Borel, not by Heine. In any case, named theorems, whether named correctly or incorrectly, are often central to a given branch of mathematics, or a monument of human accomplishment in their own right. The proof of Pythagoras' Theorem must rank with Beethoven's 9th Symphony, Shakespeare's Hamlet, and Botticelli's Venus as among the supreme achievements of human history.

These named theorems are generally few and far between—in mathematical textbooks and articles, most results carry with them no grand title, no monumental weight. Hardy writes that 'the noblest ambition is that of leaving behind something of permanent value', but the sobering truth is that the work of the mathematician is rarely that glamorous. Most theorems that the reader will come across are, in the grand scheme of events, small, insignificant, and destined to be forgotten by most, if not all, subsequent mathematicians. But the effect of those theorems is incalculably diffusive: for the growing good of mathematics is dependent on unhistoric works; and that you and I can discover such elegance and beauty in the realm of that subject is half owing to those unnamed mathematicians, who lived faithfully a hidden life, and rest in unvisited tombs.



Breaking Chaos

Nicholas Hayes

I awake in a cold sweat,
my eyes tracking the lines traced by these strange attractors:
fractal hypnosis.

In the heart of the chaotic realm,
where order emerges from tumultuous dance,
precision symphonies swirl around me in
deterministic confusion,
predictable yet intractable.

I navigate the labyrinth,
each step calculated,
each step perturbing.
Yet as I grasp the threads of order—
they sand-slip through my fingers,
leaving me stranded, no steady-state returnable
from these minor deviations at the crux of
instability.

A solitary traveler, I find
rhythm in the madness,
solace in patterned diffusion.
Previous voyages—commanded by
stochastic waters at the helm—
left me journey-full and directionless,

supreme randomness restricting my expectations,
a posteriori knowledge synthetically estimated.

I trek on, straining to spot trails of
forbearing explorers.
I whisper their names softly:

Lyapunov, Mandelbrot, Lorenz,
Feigenbaum, Turing, Libchaber,
Kuznetzov, Shaw, Strogatz.

They whisper back:

There is still beauty to be found.
Go forth, unravel the tapestry.

Is Visual Art Mathematical?

Siddiq Islam

Everyone loves to connect mathematics with music. It is a well-established concept by now. Musicians' brains are compared to mathematicians' brains. We revel in Bach's meticulous canons, augmentations, and retrogrades, which can be thought of like translations, stretches, and reflections as though the music were on a Cartesian graph (for instance in his *Crab Canon*). There is even a 'math rock' genre (although unconventional). While music is viewed as having inherently mathematical aspects, the opposite is often said of visual art. Drawing, painting and sculpture are thought of as the antithesis of rigid mathematics. They capture fluid beauty and human emotions that could not be defined by equations. This leads to the question: could visual art be linked to mathematics the same way music is?

A link that may come to mind is the golden ratio. Dubbed in Renaissance times 'the Divine Proportion', it is supposedly exhibited in artworks such as Da Vinci's *Mona Lisa* and *The Last Supper*, as well as many others. Modern photographers might use a golden spiral to compose their pictures and legend has it the most beautiful faces have eyes, noses and mouths aligned in this magical ratio. Even galaxies and shells grow in 'golden spirals'. Unfortunately, the beauty of the golden ratio is probably only a mathematical one. Online, we find golden rectangles and spirals superimposed onto Renaissance paintings quite crudely and seemingly at random. As for the shells and galaxies, they can produce logarithmic spirals, but are never in the exact ratio (and are in fact usually quite far off). Photographers may use it in their photographs, but so many beautiful pictures do not require mathematical levels of precision and people certainly do not need to conform to such a bizarre beauty standard in order to look good! Whilst Da Vinci did know about the golden ratio and used it in his lovely mathematical diagrams, he did not need it for his art. So, can artists do without mathematics, then?

When looking at medieval art, we find it disproportionate and flat, as though everyone is facing the wrong way. By the Renaissance, artists were using a trick to aid composition still taught in art class today, which is to pick a vanishing point at infinity where all lines converge. This tackles the foreshortening effect that the farther things get from the viewer, the smaller they appear (Figure 1).

Thus, geometry can help us depict things more realistically. Take another example. A wall in a room is a rectangle made of straight sides, so the corners should appear as right angles. However, when we look at the corners of a room, the angles do not appear to be 90 degrees, but rather slightly larger (Figure 2).

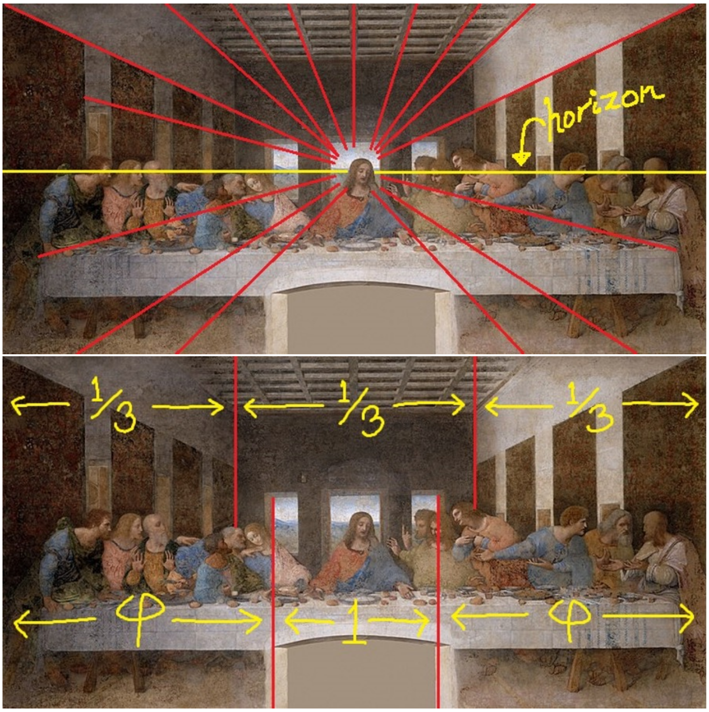


Figure 1. Jesus's face is placed at the vanishing point in *The Last Supper*, so it becomes the centre of focus. Everyone else's eyes converge near the horizon line. Photographers use the rule of thirds trick more often than the golden ratio. Although both ratios appear to fit Da Vinci's painting, it is not evident that he did this on purpose.

Figure 2. Not a right angle when looked at.

Perspective is something our brains deal with constantly without us realising. The section of wall directly in front of us is closer, so it appears larger than the section of wall in the corners of the room. To our eyes, the wall starts small in the corners of the room, gets larger in the middle, and shrinks again as it approaches the next corner. Our brains fill in the gaps to tell us the wall is a rectangle.

The phenomena described can be thought of as a consequence of our projection of a cuboid room onto our sphere of vision. When mapping the edges of the room onto our sphere of vision, the corners must add to 360 degrees, which three 90-degree angles cannot do, so the lines instead must curve and stretch (Figure 3).

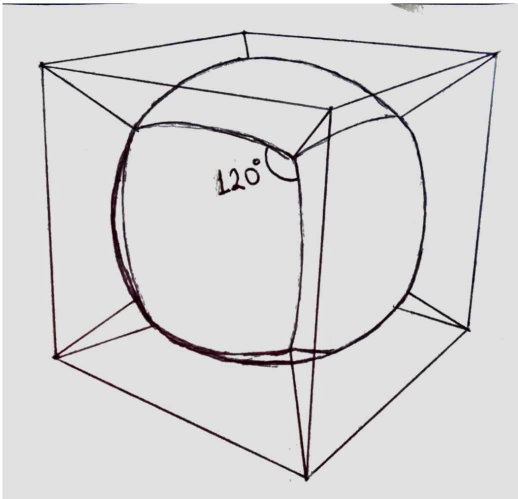


Figure 3. Projection of cube room onto sphere.

This is noticed when taking a photograph with people near the edges. Their faces become distorted as a result of rectilinear projection, the correction that a wide-angle lens makes to help things look straighter. An alternative effect occurs when using a fish-eye lens or taking a panoramic photo. This kind of image is a curvilinear projection: everything is the right proportion, but straight lines appear curved.

The bending of our visible world is not a problem most of the time—it only occurs when we are trying to capture things very far apart in our vision—but realist artists should be aware of such optical paradoxes nonetheless (Figure 4). Once again, understanding geometry can help us create more realistic images.

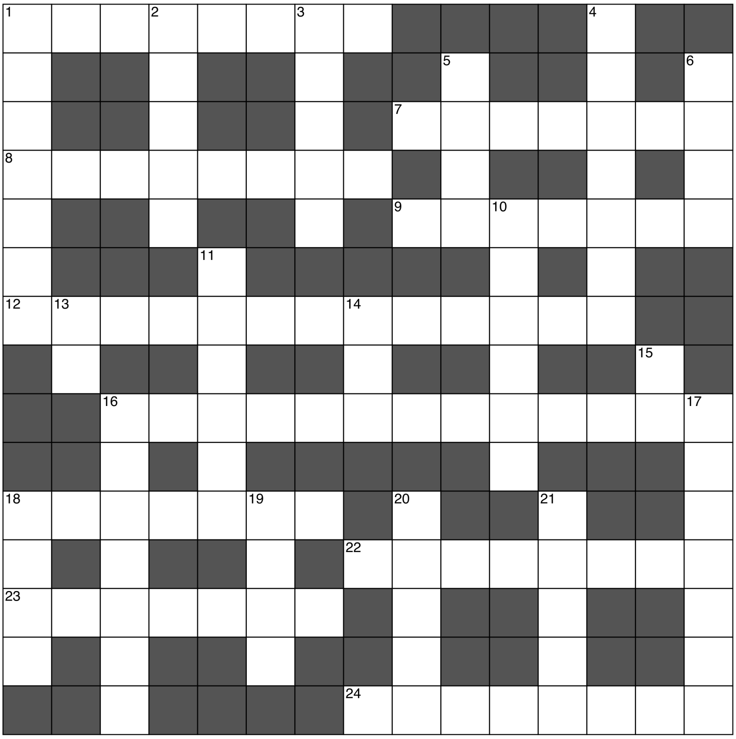
Figure 4. Rectilinear versus curvilinear projection. In the top image, the rectilinear camera lens corrects the distortion. Everything in the image looks straight, but near the corners, the image has been stretched, whereas in the second image, everything is the right size, but straight lines appear curved. The bottom image cannot be created standing in one spot. It is a rolling snapshot where everything appears to face forwards. The bookshelf on the right exhibits clearest difference. This is the flat perspective seen in medieval artwork that the Renaissance painters dismissed with their methods of perspective.

What of abstract, wavering paintings that disregard dimension? Take Pollock's *Birth* or Picasso's *Weeping Woman*. These painters cannot benefit from the above tricks. I would argue, however, that they still must have a sense of space. They must leave enough room for the eyes of an ambiguous monster even if they are not halfway (or φ-way) up the face, and while the colour and texture of individual brushstrokes might be out of their control, the shapes and forms they represent are mathematical objects in 2D space (or 3D space for sculptors), much in the same way that a composer cannot control the texture of instruments but can arrange notes to perfection on the stave.

Art is distinct from mathematics and the abstract concepts it often symbolises—whether love, conflict or sorrow—are hard to describe with numerical formulas, but the media through which visual artists communicate requires mathematical knowledge and can benefit particularly from geometry.

Cryptic Crossword

Niphredil



ACROSS

1 Sum limit necessary? (8)

7 10 = 2 (4,3)

8 Quote in a mess to be solved (8)

9 Spooner's unhasty solution for set (7)

12 Raises without former partner makes one's patient (13)

16 Friendly state of 3-D shape (8,5)

18 Scare fit meat is on or off? (7)

22 17D's opposite, swapping sigma for gamma (8)

23 Mean, as standard (7)

24 ... of dance steps, or numbers? (8)

DOWN

1 Opposite in poetry (7)

2 On par (5)

3 Taxi's end takes love to my principle (5)

4 Elsewhere places the result (7)

5 Open or closed, for ticket-holders? (4)

6 Done complicated part of graph (4)

10 Angle on funny toe bus (6)

11 Warned of dodgy function dealers stirring leaven (1,5)

13 Eleven's letter? (2)

14 X marks the spot on the number line (3)

15 Sounds tart (2)

16 Big task forecast (7)

17 Detective, hit grass and run away (7)

18 Two as leaning (4)

19 Scrappy rags give inputs (4)

20 Numbers on the radio compel (5)

21 Thick, like the rationals in the reals? (5)

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Solutions to the crossword. Across. 1. INTEGRAL. 7. BASETWO. 8. EQUATION. 9. CLOSURE. 12. EXPONENTIATES. 16. PLATONICSOLID. 18. BOOLEAN. 22. CONVERSE. 23. AVERAGE. 24. SEQUENCE. Down. 1. INVERSE. 2. EQUAL. 3. AXIOM. 4. OUTPUTS. 5. BALL. 6. NODE. 10. OBTUSE. 11. VNEALE. 13. XI. 14. TEN. 15. PI. 16. PROJECT. 17. DIVERGE. 18. BIAS. 19. ARGS. 20. FORCE. 21. DENSE.

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